

# Cellularity of hermitian $K$ -theory and Witt-theory

Oliver Röndigs, Markus Spitzweck, Paul Arne Østvær

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## Abstract

Hermitian  $K$ -theory and Witt-theory are cellular in the sense of stable motivic homotopy theory over any base scheme without points of characteristic two.

## 1 Introduction

The notion of a cellular object in motivic homotopy theory is intrinsically linked to the geometry of motivic spheres  $S^{p,q}$  [4]. Suppose the smooth scheme  $X$  admits a filtration by closed subschemes

$$\emptyset \subset X_0 \subset \cdots \subset X_{n-1} \subset X_n = X,$$

where  $X_i \setminus X_{i-1}$  is a disjoint union of affine spaces  $\mathbf{A}^{n_{ij}}$ . Examples of such filtrations arise in the context of Białynicki-Birula decompositions for  $\mathbf{G}_m$ -action on smooth projective varieties [2], cf. [3] for a more recent implementation. By homotopy purity [7, Theorem 3.2.23] for Thom spaces of normal bundles of closed embeddings, there is a homotopy cofiber sequence

$$X \setminus X_i \longrightarrow X \setminus X_{i-1} \longrightarrow \mathbf{Th}(\mathcal{N}_i).$$

By assumption the normal bundle  $\mathcal{N}_i$  is trivial. Thus the splitting  $\mathbf{Th}(\mathcal{N}_i) \cong \bigvee_j S^{2n_{ij}, n_{ij}}$  and the two-out-of-three property for stably cellular objects [4, Lemma 2.5] imply inductively that  $X$  is stably cellular in the sense of [4, Definition 2.10].

In this paper we employ a similar strategy to prove cellularity for Thom spaces of direct sums of tautological symplectic bundles over quaternionic Grassmannians. This allows us to show cellularity of the motivic spectra representing hermitian  $K$ -theory and Witt-theory [5]. By a base scheme we mean any regular noetherian separated scheme of finite Krull dimension.

**Theorem 1.1.** *Suppose all points on the base scheme have residue characteristic unequal to two. Then hermitian  $K$ -theory  $\mathbf{KQ}$  and Witt-theory  $\mathbf{KW}$  are cellular motivic spectra.*

For a related antecedent result showing cellularity of algebraic  $K$ -theory, see [4, Theorem 6.2]. The proof of Theorem 1.1 exploits the geometry of quaternionic Grassmannians and the explicit model for hermitian  $K$ -theory from [9].

Recent applications of  $\mathbf{KQ}$  and  $\mathbf{KW}$  concern computations of stable homotopy groups of motivic spheres [6], [8], [12], and a proof of the Milnor conjecture on quadratic forms [11]. For cellular motivic spectra one has the powerful fact that stable motivic weak equivalences are detected by  $\pi_{*,*}$ -isomorphisms [4, Corollary 7.2]. Our main motivation for proving Theorem 1.1 is that it is being used in the computation of the slices of  $\mathbf{KQ}$  in [12, Theorem 2.14]. In terms of motivic cohomology with integral and mod-2 coefficients, the result is

$$s_q(\mathbf{KQ}) \cong \begin{cases} \Sigma^{2q,q} \mathbf{MZ} \vee \bigvee_{i < \frac{q}{2}} \Sigma^{2i+q,q} \mathbf{MZ}/2 & q \text{ even} \\ \bigvee_{i < \frac{q+1}{2}} \Sigma^{2i+q,q} \mathbf{MZ}/2 & q \text{ odd.} \end{cases}$$

In turn, this is an essential ingredient in our proof of Morel's  $\pi_1$ -conjecture in [12]. It is an interesting problem to make sense of Theorem 1.1 without any assumptions on the points of the base scheme.

This short paper is organized into Section 2 on basic properties of motivic cellular spectra, Section 3 on the geometry of quaternionic Grassmannians, and Section 4 on hermitian  $K$ -theory and Witt-theory.

## 2 Cellular objects

The subcategory of cellular spectra in the motivic stable homotopy category is the smallest full localizing subcategory that contains all suspensions of the sphere spectrum, cf. [4, §2.8]. For our purposes it suffices to know four basic facts about cellular motivic spectra. First we recall part (3) of Definition 2.1 in [4].

**Lemma 2.1.** *The homotopy colimit of a diagram of cellular motivic spectra is cellular.*

The second fact is a specialization of [4, Lemma 2.4].

**Lemma 2.2.** *Let  $E$  be a motivic spectrum and let  $p, q$  be integers. Then  $E$  is cellular if and only if its  $(p, q)$ -suspension  $\Sigma^{p,q}E$  is cellular.*

The third fact is a specialization of [4, Lemma 2.5].

**Lemma 2.3.** *If  $E \rightarrow F \rightarrow G$  is a homotopy cofiber sequence of motivic spectra such that any two of  $E, F$ , and  $G$  are cellular, then so is the third.*

Finally, we recall Lemma 3.2 in [4].

**Lemma 2.4.** *If  $E_i$  is a cellular motivic spectrum for all  $i \in I$ , then  $\coprod_{i \in I} E_i$  is cellular.*

## 3 Quaternionic Grassmannians

The quaternionic Grassmannian  $\mathbf{HGr}(r, n)$  is the open subscheme of the ordinary Grassmannian  $\mathbf{Gr}(2r, 2n)$  parametrizing  $2r$ -dimensional subspaces of the trivial vector bundle  $\mathcal{O}^{\oplus 2n}$  on which the standard symplectic form is nondegenerate. It is smooth affine of dimension  $4r(n - r)$  over the base scheme. Let  $\mathcal{U}_{r,n}$  be short for the tautological symplectic subbundle of rank  $2r$  on  $\mathbf{HGr}(r, n)$ . It is the restriction to  $\mathbf{HGr}(r, n)$  of the tautological subbundle of  $\mathbf{Gr}(2r, 2n)$  together with the restriction to  $\mathcal{U}_{r,n}$  of the standard symplectic form on  $\mathcal{O}^{\oplus 2n}$ .

More generally, to every symplectic bundle  $(\mathcal{E}, \phi)$  one associates the quaternionic Grassmannian  $\mathbf{HGr}(r, \mathcal{E}, \phi)$ ; it is the open subscheme of the Grassmannian  $\mathbf{Gr}(2r, \mathcal{E})$  parametrizing  $2r$ -dimensional subspaces of the fibers of  $\mathcal{E}$  on which  $\phi$  is nondegenerate. Associated to the trivial rank  $2n - 2$  symplectic bundle  $(\mathcal{E}, \psi)$  is the bundle  $\mathcal{F} = \mathcal{O} \oplus \mathcal{E} \oplus \mathcal{O}$  equipped with the direct sum of  $\psi$  and the hyperbolic symplectic form, i.e.,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & \psi & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

For simplicity we write  $\mathbf{HGr}(\mathcal{E})$  for  $\mathbf{HGr}(r, \mathcal{E}, \psi)$  and likewise for  $\mathcal{F}$ .

The normal bundle  $N$  of the embedding  $\mathbf{HGr}(\mathcal{E}) \subset \mathbf{HGr}(\mathcal{F})$  is the tensor product  $\mathcal{U}_{\mathcal{E}}^{\vee} \otimes \mathcal{O}^{\oplus 2}$  for the dual of the tautological symplectic subbundle of rank  $2r$  on  $\mathbf{HGr}(\mathcal{E})$ . Theorem 4.1 in [10] shows that  $N$  is naturally isomorphic to an open subscheme of  $\mathbf{Gr}(2r, \mathcal{F})$  and there is a decomposition  $N = N^+ \oplus N^-$ ; here,  $N^+ = \mathbf{HGr}(\mathcal{F}) \cap \mathbf{Gr}(2r, \mathcal{O} \oplus \mathcal{E})$  and  $N^- = \mathbf{HGr}(\mathcal{F}) \cap \mathbf{Gr}(2r, \mathcal{E} \oplus \mathcal{O})$  have

intersection  $\mathbf{HGr}(\mathcal{E})$ . Thus there are natural vector bundle isomorphisms  $N^+ \cong N^- \cong \mathcal{U}_{r,n-1}$  and the normal bundle  $\mathcal{N}$  of  $N^+$  in  $\mathbf{HGr}(\mathcal{F})$  is isomorphic to  $\pi_+^* \mathcal{U}_{r,n-1}$  for the bundle projection  $\pi_+ : N^+ \rightarrow \mathbf{HGr}(\mathcal{E})$ . Moreover, there is a vector bundle isomorphism between the restriction  $\mathcal{U}_{r,n}|_{N^+}$  of  $\mathcal{U}_{r,n}$  to  $N^+$  and  $\pi_+^* \mathcal{U}_{r,n-1}$ . For  $r \leq n-1$ , let  $Y$  denote the complement of  $N^+$  in  $\mathbf{HGr}(\mathcal{F})$  [10, (5.1)].

**Proposition 3.1.** *For  $m \geq 0$  the suspension spectrum of the Thom space of the vector bundle  $\mathcal{U}_{r,n}^{\oplus m}$  on  $\mathbf{HGr}(r, n)$  is a finite cellular spectrum. In particular,  $\Sigma^\infty \mathbf{HGr}(r, n)_+$  is a cellular spectrum.*

*Proof.* The proof proceeds by a double induction argument on  $r$  and  $n \geq r$ . The base cases  $\mathbf{HGr}(0, n)$  and  $\mathbf{HGr}(n, n)$  are clear, so we may assume  $0 < r < n$ . Define the motivic space  $Z$  by the homotopy cofiber sequence

$$\mathbf{Th}(\mathcal{U}_{r,n}^{\oplus m}|Y) \longrightarrow \mathbf{Th}(\mathcal{U}_{r,n}^{\oplus m}) \longrightarrow Z. \quad (1)$$

According to [13, Lemma 3.5] there is a canonical isomorphism in the motivic homotopy category

$$Z \cong \mathbf{Th}(\mathcal{U}_{r,n}^{\oplus m}|N^+ \oplus \mathcal{N}).$$

Using the above we note  $\mathcal{U}_{r,n}^{\oplus m}|N^+ \oplus \mathcal{N} \cong \pi_+^* \mathcal{U}_{r,n-1}^{\oplus(m+1)}$  and hence there are canonical isomorphisms

$$Z \cong \mathbf{Th}(\pi_+^* \mathcal{U}_{r,n-1}^{\oplus(m+1)}) \cong \mathbf{Th}(\mathcal{U}_{r,n-1}^{\oplus(m+1)}).$$

By induction hypothesis  $\Sigma^\infty Z$  is a finite cellular spectrum. Thus Lemma 2.3 and (1) reduce the proof to showing that  $\Sigma^\infty \mathbf{Th}(\mathcal{U}_{r,n}^{\oplus m}|Y)$  is a finite cellular spectrum. To this end we recall parts of Theorem 5.1 in [10]: There exists maps

$$Y \xleftarrow{g_1} Y_1 \xleftarrow{g_2} Y_2 \xrightarrow{q} \mathbf{HGr}(r-1, \mathcal{E}, \psi),$$

where  $g_i$  and  $q$  are Zariski locally trivial torsors over vector bundles of rank  $2r-i$  and  $4n-3$ , respectively. Moreover,  $g_2^* g_1^* \mathcal{U}_{r,n}$  is isomorphic to  $\mathcal{O}_{Y_2}^2 \oplus q^* \mathcal{U}_{r-1,n}$ . Invoking [7, §3.2, Example 2.3] this implies the canonical isomorphisms

$$\Sigma^\infty \mathbf{Th}(\mathcal{U}_{r,n}^{\oplus m}|Y) \cong \Sigma^\infty \mathbf{Th}(g_2^* g_1^* \mathcal{U}_{r,n}^{\oplus m}|Y) \cong \Sigma^\infty \mathbf{Th}(\mathcal{O}_{Y_2}^{2m} \oplus q^* \mathcal{U}_{r-1,n}^{\oplus m}) \cong \Sigma^{2m,m} \Sigma^\infty \mathbf{Th}(\mathcal{U}_{r-1,n}^{\oplus m}).$$

Here, the suspension spectrum of  $\mathbf{Th}(\mathcal{U}_{r-1,n}^{\oplus m})$  is finite cellular by the induction hypothesis. This finishes the proof using Lemma 2.2.  $\square$

## 4 Hermitian $K$ -theory and Witt-theory

In this section we finish the proof of Theorem 1.1 stated in the introduction.

The quaternionic plane  $\mathbf{HP}^1$  is the first quaternionic Grassmannian  $\mathbf{HGr}(1, 2)$ . In the pointed unstable motivic homotopy category,  $(\mathbf{HP}^1, x_0)$  is isomorphic to the two-fold smash product of the Tate object  $T \equiv \mathbf{A}^1/\mathbf{A}^1 \setminus \{0\}$ . It follows that the  $\mathbf{A}^1$ -mapping cone  $\mathbf{HP}^{1+}$  of the rational point  $x_0 : S \rightarrow \mathbf{HP}^1$  is isomorphic to  $T^{\wedge 2}$ . Hence the stable homotopy category of  $\mathbf{HP}^{1+}$ -spectra is equivalent to the standard model for the stable motivic homotopy category [9, Theorem 12.1].

Theorem 12.3 in [9] shows there is an isomorphism between hermitian  $K$ -theory  $\mathbf{KQ}$  and an  $\mathbf{HP}^{1+}$ -spectrum  $\mathbf{BO}_{2n}^{\text{geom}}$ . For  $n$  odd,  $\mathbf{BO}_{2n}^{\text{geom}} = \mathbf{Z} \times \mathbf{HGr}$  [9, (12.5)]. Here  $\mathbf{HGr}$  denotes the infinite quaternionic Grassmannian, i.e., the sequential colimit

$$\text{colim}_n \mathbf{HGr}(n, 2n).$$

We note that the transition maps in the colimit are defined in [9, (8.1)]. The motivic space  $\mathbf{Z} \times \mathbf{HGr}$  is pointed by  $(0, \mathbf{HGr}(0, 0))$ . Thus  $\mathbf{KQ}$  is isomorphic to the homotopy colimit

$$\operatorname{hocolim}_{n \text{ odd}} \Sigma^{4n, 2n} \Sigma^\infty \mathbf{Z} \times \mathbf{HGr}. \quad (2)$$

It remains to show cellularity of (2). Note that  $\Sigma^\infty \mathbf{Z} \times \mathbf{HGr}$  is a homotopy colimit of cellular spectra by Lemma 2.4 and Proposition 3.1. It follows that  $\Sigma^{4n, 2n} \Sigma^\infty \mathbf{Z} \times \mathbf{HGr}$  is cellular according to Lemmas 2.1 and 2.2. We conclude the proof for  $\mathbf{KQ}$  by applying Lemma 2.1.

Cellularity of  $\mathbf{KW}$  follows from that of  $\mathbf{KQ}$  via Lemma 2.1 and the description of  $\mathbf{KW}$  as the homotopy colimit of the diagram

$$\mathbf{KQ} \xrightarrow{\eta} \Sigma^{-1, -1} \mathbf{KQ} \xrightarrow{\Sigma^{-1, -1} \eta} \Sigma^{-2, -2} \mathbf{KQ} \xrightarrow{\Sigma^{-2, -2} \eta} \dots$$

given in [1, Theorem 6.5]. Here,  $\eta$  is the first stable Hopf map induced by the canonical map  $\mathbf{A}^2 \setminus \{0\} \rightarrow \mathbf{P}^1$ .

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Institut für Mathematik, Universität Osnabrück, Germany.  
e-mail: oliver.roendigs@uni-osnabrueck.de

Institut für Mathematik, Universität Osnabrück, Germany.  
e-mail: markus.spitzweck@uni-osnabrueck.de

Department of Mathematics, University of Oslo, Norway.  
e-mail: paularne@math.uio.no